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Casimir energy for a massive fermionic quantum field with a spherical boundary

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Abstract. The vacuum energies corresponding to massive Dirac fields with the boundary conditions of the MIT bag model are obtained. The calculations are carried out with the fields occupying the regions inside and outside the bag, separately. The renormalization procedure for each of the situations is studied in detail, in particular the differences occurring with respect to the case when the field extends over the whole space. The final result contains several constants that undergo renormalization and can be determined experimentally only. The non-trivial finite parts which appear in the massive case are found exactly, providing a precise determination of the complete, renormalized zero-point energy in the fermionic case. The vacuum energy behaves as an inverse power of the mass, for large mass of the field.

1. Introduction

The first modern calculation of the vacuum energy density of a quantum field in the presence of boundaries is almost 50 years old. As is well known, it is due to Casimir [1]. Its first measurable consequence was the attraction in an electromagnetic vacuum of two neutral, infinitely conducting plates (thereafter called the Casimir effect, see for instance [2]). Previously, Casimir and Polder [3] had explained the attraction of two neutral bodies in terms of a retarded van der Waals effect. Later, dielectric properties of the materials considered were taken into account in the more ambitious Lifshitz theory [4]. However, Casimir [1] was the first to perform a genuine quantum field theoretical calculation using the concept of zero-point energy (whose physical relevance was somewhat unclear at that time). The treatment of the divergences resulting from the infinitely many degrees of freedom was (and still is) the most difficult aspect. Calculations of the vacuum energy have attracted the interest of many scientists, because it turns out that, in different contexts, the inclusion of quantum fluctuations about semiclassical configurations is essential. On the other hand, spherically symmetrical situations are very important for practical applications. The calculations involved are certainly much more complicated than in the case of systems with plane boundaries.

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Having found an attractive force between parallel plates due to the vacuum energy [1], the hope was that the same would be true for a spherically symmetric situation. This led Casimir to the idea that the force stabilizing a classical electron arises from the zero-point energy of the electromagnetic field within and without a perfectly conducting spherical shell [5]. Unfortunately, as Boyer [6] first showed, for this geometry the stress is repulsive [7, 8]. It is known nowadays that the Casimir energy depends strongly on the geometry and dimension of the spacetime and also on the boundary conditions imposed. This is a very active field of research (see, for instance, [9, 10]). Let us mention, in the context of the spherical Casimir effect, the analysis of its dimensional dependence presented in [11, 12].

More recently, the zero-point energy has received considerable attention in the context of the bag model [13–22] and chiral bag model [23–29]. In these systems, quarks and gluons are free inside the bag, but are absolutely confined to it, being unable to cross the boundary surface. This is imposed, mathematically, by appropriate boundary conditions. The sum of the mesonic, valence-quark and vacuum-quark contributions to the baryonic number have been found to be independent of the bag radius and of the pion field strength, being the vacuum-quark contributions—which are analogues of the Casimir effect in QED—essential for the calculation of baryonic observables. The issue of regularization in this model is quite non-trivial. It happens that, under specific circumstances, different regularization procedures can yield different results and real physical problems arise in the calculation of quark-vacuum contributions to some barionic observables, such as the energy itself.

The calculation of the Casimir energy for massless fermions in the interior of a spherical bag was already considered 20 years ago [15]. However, in this first attempt only the divergent terms were isolated. In a reconsideration of the issue, Milton also retained the finite terms, and introducing suitable phenomenological terms (contact terms) and renormalizing the respective constants, he was able to obtain a finite Casimir energy [19]. Later, the point of view was taken that including the exterior modes makes physical sense for the vacuum [30]. The pertinent calculation was carried out in [21]. In this case, a mutual cancellation (which can be termed as 'natural') of the divergences of the inner and outer spaces occurs. As a result, finite zero-point energies are obtained. Recently this idea has been revived in [23]. It has been argued that at high enough energies one expects QCD to show a phase transition to an unconfined plasma of quarks and gluons and, for that reason, one has to allow for high-energy quarks living in the exterior region. Finally, still for the massless field, finite-temperature effects were taken into account in [17, 28].

An obvious generalization of the above considerations is to try to extend them to fields of non-vanishing mass. As a first step, in the spirit of Bender and Hays [15], the divergences were determined in [22] and have been discussed in the framework of the field-theoretical bag model of Creutz and of Friedberg and Lee [31]. Alternatively, as already mentioned there, one can choose to introduce all the surface and curvature tensions which appear—with divergent factors—in the Casimir energy, from the outset, with finite coefficients, and consider the divergent contributions as being absorbed into their renormalization. This rather pragmatic viewpoint has the one taken in [32] where, using the proper time formalism, it has been demonstrated that, for a spherical bag, one needs at most the following contact terms

$$E_{\text{class}} = pV + \sigma S + FR + k + \frac{h}{R}.$$

As already emphasized there, the parameters of the 'classical' phenomenological energy E_{class} are to be determined from the experiments; they cannot be calculated within the confines of the bag model. The situation is very reminiscent of what happens in quantum field theory in curved spacetime. In fact, in that context the classical system is the gravitational field and, in order to renormalize the energy–momentum tensor of quantum matter, one needs to use a suitable general Lagrangian for the gravitational field [33]. All constants appearing in this generalized Lagrangian are to be determined experimentally and *not* within the quantum field theory model in the curved background. In our context of the bag, the classical part is represented by a model for the surface and the interpretation of the contact terms above is much the same as the one described in the context of the gravitational Lagrangian. From now on this will be our standpoint in the considerations that follow.

As is clear from the above argument, for the massive fermionic quantum field in the bag there is no analysis extending beyond the isolation of the divergent terms. In contrast, it is the main aim of the present analysis to also retain the finite part of the energy. As opposed to the massless case, this energy depends in a nontrivial way on the dimensionless parameter mR, m being the mass of the field and R the radius of the bag. This explicit dependence will be determined here for the first time.

In most of the papers mentioned above a Green function approach has been used in order to calculate the zero-point energy. An exception is [32], where, in the general setting of an ultrastatic spacetime with or without boundaries, a systematic procedure which makes use of zeta-function regularization was developed. In this approach, a knowledge of the zeta function of the operator associated with the field equation together with (eventually) some appropriate boundary conditions is needed. Recently, a detailed description of how to obtain the zeta function for a massive scalar field inside a ball satisfying Dirichlet or Robin boundary conditions has been given elsewhere by the authors of this work [34, 35]. An analytical continuation to the whole complex plane was obtained there and subsequently applied to the computation of an arbitrary number of heat-kernel coefficients. In ensuing papers [36, 37] the functional determinant was considered too and, furthermore, the method has also been applied to spinors [38, 39] and p-forms [40–42] (for a different approach see [43]). All the above considerations are purely analytical and quite precise. In order to obtain explicit values for the Casimir energy, however, a numerical evaluation of an integral and a sum was necessary. This has been achieved in different cases, in particular for the massless scalar field and the electromagnetic field [44], partly re-obtaining previous results.

To finish this description of recent previous work, let us mention that in [45] we have investigated the case of a massive scalar field in the bag. We have discussed there how, for the case of a massive field—already for a *scalar* one—non-trivial finite parts which depend on an adimensional variable involving the mass are present, that need to be properly renormalized, in order to obtain the corresponding zero-point energy. In this paper we shall extend our analysis to the case of Dirac fields, thus generalizing our considerations to a situation which approaches very much the conditions of a realistic MIT bag model.

The organization of the paper is as follows. We shall rely on our previous work (dealing with the bosonic case) for a precise description of the method employed—which was given there in full detail [45]—as well as for the particular formulae that are needed in the subsequent study of the zeta function of the problem we consider here. We felt that to repeat all this here would not be justified. Consequently, in section 2 we proceed immediately with the specific description of the model for the case of Dirac fields inside the bag with boundary conditions corresponding to the MIT bag. Starting from the Dirac equation and imposing the boundary conditions we will derive an eigenvalue equation in terms of Bessel functions. This will be the basic equation to solve, which we shall do in the same section for the interior of the bag. In section 3 we will describe the renormalization scheme used in the model. Section 4 contains the analogous treatment for the region exterior to the bag and for the whole space. Adding up the interior and the exterior contributions, we will see how the divergences cancel exactly among themselves, as well as the influence of this cancellation on the compulsory renormalization process. It turns out that important differences with respect to the non-fermionic case appear concerning this issue, although we shall argue that, in the end, they will not substantially effect the interpretation of the physical results. Section 5 is devoted to conclusions. The appendices contain some hints and technical details that have been used for the derivation of the zeta function (appendix A) and a full list of the constituents that build up the subtraction terms in the decomposition of the zeta function, an essential (although rather technical) step in our method (appendix B).

2. Fermions inside the bag

The first task is to derive the energy eigenvalue equations for a Dirac spinor subject to the MIT bag boundary conditions. The setting we consider first is the Dirac spinor inside a spherically symmetric bag confined to it by the appropriate boundary conditions. The coordinates we use are just the spherical ones, r, θ, φ, which best adapt to the form of the bag. Thus, we must solve the equation:

$$H\phi_n(\boldsymbol{r}) = E_n\phi_n(\boldsymbol{r}) \tag{2.1}$$

H being the Hamiltonian,

$$H = -\mathrm{i}\gamma^0\gamma^\alpha\partial_\alpha + \gamma^0 m \tag{2.2}$$

with the boundary conditions

$$\left[1 + \mathrm{i}\left(\frac{\boldsymbol{r}}{r}\gamma\right)\right]\phi_n|_{r=R} = 0. \tag{2.3}$$

These boundary conditions guarantee that no quark current is lost through the boundary.

The separation to be carried out in the eigenvalue equation (2.1) is rather standard and will not be given in detail here. Let $\boldsymbol{J}$ be the total angular momentum operator and K the spin projection operator. Then there exists a simultaneous set of eigenvectors of H, $\boldsymbol{J}^2$, J_3, K and the parity P. The eigenfunctions for positive eigenvalues $\kappa = j + \frac{1}{2}$ of K read

$$\phi_{jm} = \frac{A}{\sqrt{r}}\begin{pmatrix} \mathrm{i}J_{j+1}(\omega r)\Omega_{jlm}(\frac{\boldsymbol{r}}{r}) \\ -\sqrt{\frac{E-m}{E+m}}J_j(\omega r)\Omega_{jl'm}(\frac{\boldsymbol{r}}{r})\end{pmatrix} \tag{2.4}$$

whereas, for $\kappa = -(j + 1/2)$, one finds

$$\phi_{jm} = \frac{A}{\sqrt{r}}\begin{pmatrix} \mathrm{i}J_{j}(\omega r)\Omega_{jlm}(\frac{\boldsymbol{r}}{r}) \\ \sqrt{\frac{E-m}{E+m}}J_{j+1}(\omega r)\Omega_{jl'm}(\frac{\boldsymbol{r}}{r})\end{pmatrix}. \tag{2.5}$$

Here $\omega = \sqrt{E^2 - m^2}$, A is a normalization constant and $\Omega_{jlm}(\boldsymbol{r}/r)$ are the well known spinor harmonics. In order to obtain eigenfunctions of the parity operator we must set $l' = l - 1$ in (2.4) and $l' = l + 1$ in (2.5). In both cases, $j = \frac{1}{2}, \frac{3}{2}, \ldots, \infty$, and the eigenvalues are degenerate in $m = -j, \ldots, +j$.

Imposing the boundary conditions (2.3) on the solutions (2.4) and (2.5), respectively, one easily finds the corresponding implicit eigenvalue equation. For $\kappa > 0$, it reads

$$\sqrt{\frac{E+m}{E-m}}J_{j+1}(\omega R) + J_j(\omega R) = 0 \tag{2.6}$$

and for $\kappa < 0$, in turn,

$$J_j(\omega R) - \sqrt{\frac{E-m}{E+m}}J_{j+1}(\omega R) = 0. \tag{2.7}$$

Regretfully, it is not possible to find an explicit solution of equations (2.6) and (2.7), but as we have shown in our previous paper for the case of the scalar field—and will describe below for the spinor field—the information displayed in (2.6) and (2.7) is already enough for the calculation of the ground-state energy for massive spinors in the bag.

The regularization of this ground-state energy will be performed by using the zeta-function method. In short, we consider

$$E_0(s) = -\tfrac{1}{2}\sum_k (E_k^2)^{1/2-s}\mu^{2s} \qquad \mathrm{Re}\, s > s_0 = 2$$
$$= -\tfrac{1}{2}\zeta^{(\mathrm{int})}(s-\tfrac{1}{2})\mu^{2s} \tag{2.8}$$

and later analytically continue to the value $s = 0$ in the complex plane. Here s_0 is the abscissa of convergence of the series, μ the usual mass parameter and

$$\zeta^{(\mathrm{int})}(s) = \sum_k (E_k^2)^{-s}. \tag{2.9}$$

The power of this method lies in the well defined prescriptions and procedures that we have at hand to analytically continue the series to the rest of the complex s-plane, even when the spectrum E_k is not known explicitly (as will in fact be the case). These procedures have been developed and described in great detail in [34, 35, 45] so that we can be brief.

The zeta function in the interior space is given by

$$\zeta^{(\mathrm{int})}(s) = 2\sum_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty}(2j+1)\int_\gamma \frac{\mathrm{d}k}{2\pi \mathrm{i}}(k^2+m^2)^{-s}$$
$$\times\frac{\partial}{\partial k}\ln\left[J_j^2(kR) - J_{j+1}^2(kR) + \frac{2m}{k}J_j(kR)J_{j+1}(kR)\right]. \tag{2.10}$$

Here the factor of 2 results from taking into account particles and antiparticles. Using the method—ordinarily employed in this situation—of deforming the contour which originally encloses the singular points on the real axis, until it covers the imaginary axis, after simple manipulations we obtain the following equivalent expression for $\zeta^{(\mathrm{int})}$:

$$\zeta^{(\mathrm{int})}(s) = \frac{2\sin\pi s}{\pi}\sum_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty}(2j+1)\int_{mR/j}^{\infty}\mathrm{d}z\left[\left(\frac{zj}{R}\right)^2 - m^2\right]^{-s}$$
$$\times\frac{\partial}{\partial z}\ln\left\{z^{-2j}\left[I_j^2(zj)\left(1+\frac{1}{z^2}-\frac{2mR}{z^2 j}\right) + I_j'^2(zj)\right.\right.$$
$$\left.\left.+\frac{2R}{zj}\left(m-\frac{j}{R}\right)I_j(zj)I_j'(zj)\right]\right\}. \tag{2.11}$$

As is usual, we now split the zeta function into two parts:

$$\zeta^{(\mathrm{int})}(s) = Z_N^{(\mathrm{int})}(s) + \sum_{i=-1}^{N} A_i^{(\mathrm{int})}(s) \tag{2.12}$$

namely a regular one, $Z_N^{(\mathrm{int})}$, and a remainder that contains the contributions of the N first terms of the Bessel functions $I_\nu(k)$ as $\nu, k \to \infty$ with ν/k fixed [46]. The number N of terms that have to be subtracted is in general the minimal one necessary in order to absorb all possible divergent contributions into the ground-state energy, equation (2.8). In our case, $N = 3$. This is a general procedure, commonly applied in order to deal with such kinds of divergence. We obtain

$$Z_3^{(\text{int})}(s) = 2\frac{\sin\pi s}{\pi}\sum_{j=\frac{1}{2}}^{\infty}(2j+1)\int_{\frac{mR}{j}}^{\infty}\mathrm{d}z\left[\left(\frac{zj}{R}\right)^2 - m^2\right]^{-s}$$
$$\times\frac{\partial}{\partial z}\left\{\ln\left[I_j^2(zj)\left(1+\frac{1}{z^2}-\frac{2mR}{z^2 j}\right)+I_j'^2(zj)+\frac{2R}{zj}\left(m-\frac{j}{R}\right)I_j(zj)I_j'(zj)\right]\right.$$
$$\left.-\ln\left[\frac{\mathrm{e}^{2j\eta}(1+z^2)^{\frac{1}{2}}(1-t)}{\pi j z^2}\right]-\sum_{k=1}^{3}\frac{D_k(mR,t)}{j^k}\right\} \tag{2.13}$$

where $\eta = \sqrt{1+z^2}+\ln[z/(1+\sqrt{1+z^2})]$ and $t = 1/\sqrt{1+z^2}$. After renaming $mR = x$, the relevant polynomials are given by

$$D_1(t) = \frac{t^3}{12}+\left(x-\frac{1}{4}\right)t$$
$$D_2(t) = -\frac{t^6}{8}-\frac{t^5}{8}+\left(-\frac{x}{2}+\frac{1}{8}\right)t^4+\left(-\frac{x}{2}+\frac{1}{8}\right)t^3-\frac{t^2x^2}{2}$$
$$D_3(t) = \frac{179t^9}{576}+\frac{3t^8}{8}+\left(-\frac{23}{64}+\frac{7x}{8}\right)t^7+\left(x-\frac{1}{2}\right)t^6+\left(\frac{9}{320}-\frac{x}{4}+\frac{x^2}{2}\right)t^5$$
$$+\left(\frac{x^2}{2}+\frac{1}{8}-\frac{x}{2}\right)t^4+\left(-\frac{x}{8}+\frac{5}{192}+\frac{x^3}{3}\right)t^3. \tag{2.14}$$

The asymptotic contributions $A_i^{(\text{int})}(s)$, $i = -1,\ldots,3$, are defined as

$$A_{-1}^{(\text{int})}(s) = \frac{8\sin(\pi s)}{\pi}\sum_{j=\frac{1}{2}}^{\infty}j\left(j+\frac{1}{2}\right)\int_{mR/j}^{\infty}\left(\left(\frac{xj}{R}\right)^2-m^2\right)^{-s}\frac{\sqrt{1+x^2}-1}{x}$$
$$A_0^{(\text{int})}(s) = \frac{4\sin(\pi s)}{\pi}\sum_{j=\frac{1}{2}}^{\infty}\left(j+\frac{1}{2}\right)\int_{mR/j}^{\infty}\left(\left(\frac{xj}{R}\right)^2-m^2\right)^{-s}\frac{\partial}{\partial x}\ln\frac{\sqrt{1+x^2}(1-t)}{x^2} \tag{2.15}$$
$$A_i^{(\text{int})}(s) = \frac{4\sin(\pi s)}{\pi}\sum_{j=\frac{1}{2}}^{\infty}\left(j+\frac{1}{2}\right)\int_{mR/j}^{\infty}\left(\left(\frac{xj}{R}\right)^2-m^2\right)^{-s}\frac{\partial}{\partial x}\frac{D_i(t)}{j^i}.$$

Their small mass expansions can be conveniently represented as

$$A_{-1}^{(\text{int})}(s) = \frac{R^{2s}}{\sqrt{\pi}\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(mR)^{2k}\frac{\Gamma(k+s-\frac{1}{2})}{k+s}$$
$$\times[2\zeta(2k+2s-2,\tfrac{1}{2})+\zeta(2k+2s-1,\tfrac{1}{2})]$$
$$A_0^{(\text{int})}(s) = -\frac{R^{2s}}{\sqrt{\pi}\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(mR)^{2k}\frac{\Gamma(k+s+\frac{1}{2})}{k+s}$$
$$\times(2\zeta(2k+2s-1,\tfrac{1}{2})+\zeta(2k+2s,\tfrac{1}{2})) \tag{2.16}$$
$$A_i^{(\text{int})}(s) = -\frac{2R^{2s}}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(mR)^{2k}[2\zeta(2k+2s+i-1,\tfrac{1}{2})+\zeta(2k+2s+i,\tfrac{1}{2})]$$
$$\times\sum_{a=0}^{2i}x_{i,a}\frac{\Gamma(k+s+\frac{a+i}{2})}{\Gamma(\frac{a+i}{2})}.$$

In this expression, the $x_{i,a}$ are the coefficients of the expansion of the functions $D_i(t)$, i.e.

$$D_i(t) = \sum_{a=0}^{2i} x_{i,a} t^{a+i}. \qquad (2.17)$$

Note that here we encounter the same problem that occurred already in the scalar case. One needs a representation that is useful and valid for an (in principle) arbitrary value of m. To this end one can actually proceed in different ways, casting the final result in terms of convergent series or integrals. Our *leitmotiv* will be the following: we will always try to express the final result in terms of the formula which is more appropriate for practical evaluation (e.g. numerical, in general). This means that, sometimes, instead of having the closed convergent sums that were universally used in the scalar case, rapidly converging integrals—better suited for numerical analysis—will be preferred here.

With this aim, we note that after performing the z-integration the $A_i^{(\mathrm{int})}(s)$, for $i \geqslant 1$, can be written in the following form,

$$A_i(s) = -\frac{4m^{-2s}}{\Gamma(s)} \sum_{a=0}^{2i} \frac{x_{i,a}}{(mR)^{i+a}} \frac{\Gamma(s+(i+a)/2)}{\Gamma((i+a)/2)} \times [f(s;1+a;(i+a)/2) + \tfrac{1}{2} f(s;a;(i+a)/2)] \qquad (2.18)$$

with the definition

$$f(s;a;b) = \sum_{\nu=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \nu^a \left(1 + \left(\frac{\nu}{mR}\right)^2\right)^{-s-b}. \qquad (2.19)$$

The remaining task in this case is to calculate the $f(s;a;b)$ for the relevant values at $s = -\frac{1}{2}$. This is a systematic calculation that will be sketched in appendix A. Let us mention here just that an essential step is to use the simple recurrence:

$$f(s;a;b) = (mR)^2 [f(s;a-2;b-1) - f(s;a-2;b)]. \qquad (2.20)$$

In appendix B we give the whole list of starting terms that, in addition to the recurrence formula (2.20), are strictly necessary for obtaining explicitly all the $A_i^{(\mathrm{int})}(s)$ needed in our calculation.

3. Discussion of the renormalization

For the discussion of the renormalization let us isolate the divergent terms in the ground-state energy. By construction they are all contained in the contributions $A_i^{(\mathrm{int})}(s)$. Having their explicit form at hand (see appendices A and B) they can be given quickly. In particular, we have, for the interior part

$$\operatorname{res} A_{-1}^{(\mathrm{int})}(-\tfrac{1}{2}) = -\frac{m^4R^3}{6\pi} + \frac{m^2R}{12\pi} + \frac{7}{480\pi R}$$

$$\operatorname{res} A_{0}^{(\mathrm{int})}(-\tfrac{1}{2}) = -\frac{m^2R}{2\pi} - \frac{1}{24\pi R}$$

$$\operatorname{res} A_{1}^{(\mathrm{int})}(-\tfrac{1}{2}) = -\frac{m^3R^2}{\pi} + \frac{m^2R}{12\pi} + \frac{m}{12\pi} - \frac{1}{48\pi R}$$

$$\operatorname{res} A_{2}^{(\mathrm{int})}(-\tfrac{1}{2}) = -\frac{m^2R}{4} - m\left(\frac{1}{8} + \frac{1}{2\pi}\right) + \frac{1}{128R} + \frac{1}{24\pi R}$$

$$\operatorname{res} A_{3}^{(\mathrm{int})}(-\tfrac{1}{2}) = \frac{2m^3R^2}{3\pi} + m^2R\left(\frac{1}{4} + \frac{2}{3\pi}\right) + m\left(\frac{1}{8} + \frac{7}{20\pi}\right) - \frac{1}{128R} - \frac{97}{10\,080\pi R}$$

and, as a result,

$$\operatorname{res}\zeta^{(\mathrm{int})}(-\tfrac{1}{2}) = -\frac{1}{63\pi R} - \frac{m}{15\pi} + \frac{m^2 R}{3\pi} - \frac{m^3 R^2}{3\pi} - \frac{m^4 R^3}{6\pi}. \tag{3.1}$$

Here res denotes the residue. These terms constitute the minimal set of counterterms necessary in order to renormalize our theory.

In the scalar case one had the peculiar situation that there were no divergent contributions of the form $\sim m^3, m$ in the zeta-function description [45]—although in other regularizations they indeed appear [32]. So, in principle, one had the choice of renormalizing the associated couplings. In contrast, as seen in (3.1), for spinors the coupling constants of *all* terms appearing have to be renormalized. The minimal set of counterterms, equation (3.1), in the zeta-function scheme applied here is the same set that is found using a proper time cut-off [32].

We are led into a physical system consisting of two parts.

(1) A classical system consisting of a spherical surface ('bag') with radius R. Its energy reads

$$E_{\mathrm{class}} = pV + \sigma S + FR + k + \frac{h}{R} \tag{3.2}$$

where $V = \frac{4}{3}\pi R^3$ and $S = 4\pi R^2$ are the volume and the surface of the bag, respectively. This energy is determined by the parameters: p pressure, σ surface tension, and F, k, and h which do not have special names.

(2) A spinor quantum field $\varphi(x)$ obeying the Dirac equation and the MIT boundary conditions (2.3) on the surface. The quantum field has a ground-state energy given by E_0, equation (2.8).

It is seen that, in the limit $m \to 0$, only one divergent contribution proportional to $1/R$ survives. As a result, equation (3.2) simplifies to $E_{\mathrm{class}} = h/R$. For dimensional reasons it is clear that this is also the form of the finite part of the ground-state energy given by E_0. One thus obtains

$$E_0 = \frac{1}{126\pi R}\left(\frac{1}{s} + \ln(\mu R)^2\right) + 0.01 \times \frac{1}{R}. \tag{3.3}$$

The philosophy is now, that the complete energy of the physical system can be written as

$$E = E_{\mathrm{class}} + E_0 \tag{3.4}$$

and that the term proportional to $1/R$ can be absorbed into the definition of the phenomenological parameter of the bag model. In the example considered, the energy then reads

$$E = \frac{1}{R}\left(h_{\mathrm{ren}} + \frac{1}{126\pi}\ln(\mu R)^2\right) \tag{3.5}$$

with the definition

$$h_{\mathrm{ren}} = h + \frac{1}{126\pi R}\frac{1}{s} + 0.01 \times \frac{1}{R}. \tag{3.6}$$

This is all one can say within the confines of the bag model [32, equation (6.12)]. In particular, h_{ren} is not calculable within the model and has to be determined experimentally.

The dependence on μ has to be viewed as a remainder of the remormalization process. Milton [19] has used instead a cut-off δ arising from the non-coincidence in time of field points. Owing to the different schemes employed, there is actually no reason why the finite part in (3.3) should be equal to that obtained by Milton [19], since it varies by simply changing the value of the parameter μ.

However, once the energy E_0 is finite and once there is no remormalization ambiguity, our finite result agrees with the result of Milton. This is the case when the whole space is considered as described in section 4.

For the massive field the philosophy is the same, but one needs the full classical energy equation (3.2). First we perform a kind of minimal subtraction, where only the divergent contribution is eliminated,

$$\begin{aligned} p &\to p - \frac{m^4}{16\pi^2}\frac{1}{s} \qquad \sigma \to \sigma - \frac{m^3}{24\pi^2}\frac{1}{s} \\ F &\to F + \frac{m^2}{6\pi}\frac{1}{s} \qquad k \to k - \frac{m}{30\pi}\frac{1}{s} \\ h &\to h - \frac{1}{126\pi}\frac{1}{s}. \end{aligned} \tag{3.7}$$

As emphasized before, the quantities $\alpha = \{p, \sigma, F, k, h\}$ constitute a set of free parameters of the theory to be determined experimentally. In principle we are free to perform finite renormalizations of our choice in all of the parameters. In order to give the numerical analysis of the energy as a function of mR we are going to perform the specific finite renormalization to be described now.

First, it is possible to determine the asymptotic behaviour of the A_i for $m \to \infty$ using the results of appendices A and B. The finite pieces, not vanishing in the limit $m \to \infty$, are all of the same type as appearing in the classical energy. Our first finite renormalization is such that those pieces are cancelled. As a result, only the 'quantum contributions' are finally included, because, physically, a quantum field of infinite mass is not expected to fluctuate. The resulting A_i will be called $A_i^{(\text{ren})}$.

Concerning Z_3 we have not been able to determine analytically its complete non-vanishing behaviour for $m \to \infty$. Instead, for the numerical analysis performed, as shown in figure 1, we have constructed a numerical fit of Z_3 using a polynomial of the form

$$P(m) = \sum_{i=0}^{4} c_i m^i$$

and then we have subtracted this polynomial from Z_3. As explained above, this method is nothing else than an ulterior finite renormalization. The result will be denoted by $Z_3^{(\text{ren})}$.

Summing up, we can write the complete energy as

$$E = E_{\text{class}} + E_0^{(\text{ren})} \tag{3.8}$$

where E_{class} is defined as in (3.2) with the renormalized parameters α and $E_0^{(\text{ren})} = Z_3^{(\text{ren})} + \sum_{i=-1}^{3} A_i^{(\text{ren})}$.

Figure 1 shows the numerical analysis of the energy $E_0^{(\text{ren})}$ of the system for this specific choice of renormalization. The energy exhibits a clear minimum corresponding to a stabilizing bag radius.

4. Exterior of the bag and a model for the whole space

The analysis of the region exterior to the bag is quite similar to the one carried out for the interior region. Only some specific differences appear both in the formulae and in the results. The expression for the zeta function in the exterior region is essentially the same as that corresponding to the interior, but for the replacement of the Bessel I_j functions with

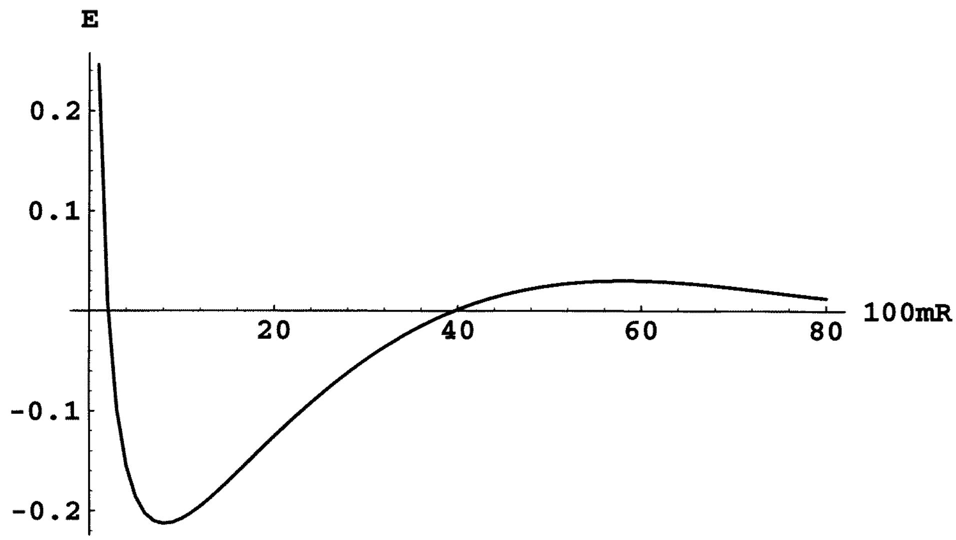

Figure 1. The energy $E_0^{(\mathrm{ren})}$ as a function of the radius for a specific choice of parameters.

Bessel K_j functions, namely

$$\zeta^{(\mathrm{ext})}(s) = \frac{2\sin\pi s}{\pi} \sum_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} (2j+1) \int_{mR/j}^{\infty} \mathrm{d}z\,((zj/R)^2 - m^2)^{-s} \times \frac{\partial}{\partial z} \ln\left[z^{2j}\left(K_j^2(zj) + K_{j+1}^2(zj) + \frac{2mR}{zj} K_j(zj) K_{j+1}(zj)\right)\right]. \quad (4.1)$$

In order to avoid volume divergences, in this expression the 'vacuum' or volume energy has already been subtracted. The splitting of the zeta function also has the same aspect as for the interior region. We have, in particular

$$A_{-1}^{(\mathrm{ext})}(s) = -A_{-1}^{(\mathrm{int})}(s)$$

$$A_0^{(\mathrm{ext})}(s) = \frac{4\sin(\pi s)}{\pi} \sum_{j=\frac{1}{2}}^{\infty} \left(j+\frac{1}{2}\right) \int_{mR/j}^{\infty} \mathrm{d}z\,((zj/R)^2 - m^2)^{-s} \frac{\partial}{\partial z} \ln\left[\frac{1+t}{t}\right]$$

and the polynomials that replace the $D_i(t)$ above are here ($x = mR$)

$$\overline{D}_1(t) = \frac{t}{4} + xt - \frac{t^3}{12}$$

$$\overline{D}_2(t) = \frac{-x^2 t^2}{2} - \frac{t^3}{8} - \frac{xt^3}{2} + \frac{t^4}{8} + \frac{xt^4}{2} + \frac{t^5}{8} - \frac{t^6}{8}$$

$$\overline{D}_3(t) = \frac{-5t^3}{192} - \frac{xt^3}{8} + \frac{x^3t^3}{3} + \frac{t^4}{8} + \frac{xt^4}{2} + \frac{x^2t^4}{2} - \frac{9t^5}{320} - \frac{xt^5}{4} - \frac{x^2t^5}{2} - \frac{t^6}{2} - xt^6 + \frac{23t^7}{64} + \frac{7xt^7}{8} + \frac{3t^8}{8} - \frac{179t^9}{576}.$$

As for the functions $A_i^{(\mathrm{ext})}(s)$, one obtains the same expressions as before, but for the replacement of the polynomials $D_i(t)$ with the corresponding polynomials $\overline{D}_i(t)$.

In principle, the same procedure as above can be applied now in order to obtain an analytical expression for the whole energy of the exterior space. Instead, we wish to restrict ourselves here to the specific changes that show up when discussing the renormalization. For that, we only have to consider the pole of the different $A_i^{(\mathrm{ext})}$. In particular, for the residues we have

$$\mathrm{res}\, A_{-1}^{(\mathrm{ext})}(-\tfrac{1}{2}) = \frac{m^4R^3}{6\pi} - \frac{m^2R}{12\pi} - \frac{7}{480\pi R} = -\,\mathrm{res}\, A_{-1}^{(\mathrm{int})}(-\tfrac{1}{2})$$
$$\mathrm{res}\, A_{0}^{(\mathrm{ext})}(-\tfrac{1}{2}) = \frac{m^2R}{2\pi} + \frac{1}{24\pi R} = -\,\mathrm{res}\, A_{0}^{(\mathrm{int})}(-\tfrac{1}{2})$$
$$\mathrm{res}\, A_{1}^{(\mathrm{ext})}(-\tfrac{1}{2}) = -\frac{m^3R^2}{\pi} - \frac{m^2R}{12\pi} + \frac{m}{12\pi} + \frac{1}{48\pi R}$$
$$\mathrm{res}\, A_{2}^{(\mathrm{ext})}(-\frac{1}{2}) = -\frac{m^2R}{4} + m\left(\frac{1}{8} - \frac{1}{2\pi}\right) + \frac{1}{128R} - \frac{1}{24\pi R}$$
$$\mathrm{res}\, A_{3}^{(\mathrm{ext})}(-\tfrac{1}{2}) = \frac{2m^3R^2}{3\pi} + m^2R\left(\frac{1}{4} - \frac{2}{3\pi}\right) - m\left(\frac{1}{8} - \frac{7}{20\pi}\right) - \frac{1}{128R} + \frac{97}{10\,080\pi R}.$$

This yields for the residue of the whole zeta function at the exterior region

$$\mathrm{res}\,\zeta^{(\mathrm{ext})}(-\tfrac{1}{2}) = \frac{1}{63\pi R} - \frac{m}{15\pi} - \frac{m^2R}{3\pi} - \frac{m^3R^2}{3\pi} + \frac{m^4R^3}{6\pi}. \tag{4.2}$$

Thus the minimal set of counterterms necessary in order to renormalize the theory in the exterior of the bag is identical to the one needed in the interior of the bag. The classical system is again described by equation (3.2).

The opposite sign of the coefficients in the divergences (3.1) and (4.2) corresponding to the odd powers of R can be easily explained by means of differential geometrical arguments, just observing that the curvature of the surface of the bag has opposite sign when looked at from the exterior or from the interior of the bag.

However, the divergences with even powers of R do not annihilate when adding up the two contributions from the two sides. In fact, for the zeta function corresponding to the whole space (internal and external to the bag) we obtain:

$$\mathrm{res}\,\zeta(-\tfrac{1}{2}) = \mathrm{res}\,\zeta^{(\mathrm{int})}(-\tfrac{1}{2}) + \mathrm{res}\,\zeta^{(\mathrm{ext})}(-\tfrac{1}{2}) = -\frac{2m}{15\pi} - \frac{2m^3R^2}{3\pi} \tag{4.3}$$

therefore, the two free parameters σ and k remain even if the whole space is considered.

The only exception is the case of the massless field, where the two (potentially) divergent contributions vanish. As a result a finite ground-state energy E_0 remains and no renormalization process is necessary. In that case our result for the energy E_0 fully agrees with the result of Milton [21],

$$E_0 = \frac{1}{R} \times 0.0204. \tag{4.4}$$

In detail, for $R = 1$, the contributions of the single constituents are summarized in tables 1 and 2; on the left for $Z_3^{(\mathrm{whole})} = Z_3^{(\mathrm{int})} + Z_3^{(\mathrm{ext})}$, on the right for $A_i^{(\mathrm{whole})} = A_i^{(\mathrm{int})} + A_i^{(\mathrm{ext})}$. In addition for $Z_3^{(\mathrm{whole})}$ the numerical value is subdivided into the single angular momenta. The contribution of $\sum_{j=\frac{7}{2}}^{\infty}$ has been obtained using the asymptotics of the integrands in equations (2.13) and (4.1), which is justified numerically.

5. Conclusions

In this paper we have studied in considerable detail a quantum field theoretical system consisting of a Dirac field with boundary conditions corresponding to those of the MIT bag

Table 1.

$j = \frac{1}{2}$	−0.028 66
$j = \frac{3}{2}$	−0.002 67
$j = \frac{5}{2}$	−0.000 88
$j = \sum_{j=\frac{7}{2}}^{\infty}$	−0.001 59

Table 2.

0	$A_{-1}^{(\mathrm{whole})}$
0	$A_{0}^{(\mathrm{whole})}$
0	$A_{1}^{(\mathrm{whole})}$
−0.015 06	$A_{2}^{(\mathrm{whole})}$
0.069 23	$A_{3}^{(\mathrm{whole})}$

model. This is the most natural continuation—in the direction towards approaching truly realistic physical systems—of previous work where only scalar fields were treated [45]. The application of our techniques can be carried out essentially in the same way as for the scalar case. Starting from the Dirac equation and imposing the boundary conditions we have derived an eigenvalue equation in terms of Bessel functions. This basic expression has then been solved, implicitly, in the regions interior and exterior to the bag surface, by using contour integration. This has yielded the corresponding zeta function in each of the two domains. Extraction of the singular part of the zeta function has also been done exactly. Adding up the contributions of the two parts, not all divergences cancel among themselves, what theoretically influences the playground of the ulterior renormalization process.

When considering a massive fermionic quantum field only in the interior or exterior of the bag, we have seen that in order to renormalize the ground-state energy E_0 we need a classical energy containing five free parameters. Adding up interior and exterior regions two of the parameters remain (for the case of non-vanishing mass). As repeatedly emphasized, these parameters cannot be fixed theoretically, but have to be numerically adjusted by means of direct comparison with the physical system described by the model [32, 45]. In this, we must confess, we are still a bit far from our final aim, in the sense that, as it stands, our model cannot be considered yet to describe a realistic physical situation. This must be left to future work, given the complexity of the proposal. In any case, we should like to point out the rigour and strict systematicity of the approach we have used here, and also its relative simplicity, if we compare it with other methods of similar strength and ambition.

Specializing our considerations to the massless field, we can compare our results with the analysis of Milton [19, 21]. Considering only the interior of the bag, we have seen that we cannot calculate the phenomenological parameter h. This issue depends very much on the regularization scheme chosen. As mentioned already, in Milton's approach divergences are of a different type. Specifically, no contact term h/R was necessary there; instead, the parameter h was calculated. However, the value of h obtained (adding up the contributions of free gluon and fermionic fields) is not in agreement with h determined from mass fits [47]. In our opinion this is no severe problem because as explained, h is not calculable within the bag model but rather fixed by the mass fits. In addition, it should be noted that perfectly acceptable fits can be made to the hadron spectrum in the bag model with the calculated Casimir energy if an additional constant force parameter, as in (3.2), is included.

When considering the massless field in the whole space no divergences at all appear and no renormalization is necessary. The finite value of E_0 obtained agrees with the value of Milton and may be contemplated as the Casimir energy of the field for this configuration.

Going beyond $m = 0$ we have determined the dependence of E_0 as a function of the dimensionless parameter mR. In doing this we have continued the analysis in [22], where only the divergent part had been determined. A notable result of our analysis, that we would like to mention, is that the Casimir energy may exhibit a clear minimum associated with a stable bag radius (see figure 1). Comparing such behaviour with the one corresponding to the scalar field, where a maximum occurred, the difference can clearly be traced back to the anticommuting nature of the spinor fields, which shows up as a sign in the definition of the ground-state energy.

Another interesting observation is that, in contrast to the case of parallel plates, the behaviour of the Casimir energy for large values of mR is not exponentially damped. Instead, as neatly observed from the representations of the $A_i(s)$ given in appendices A and B, we find a behaviour in inverse powers of the mass. This is directly connected with the non-vanishing of the extrinsic curvature at the bag.

Possible continuations of our approach go in the direction of finite temperature and finite densities, as considered already for massless fermions in [17, 28, 29]. A natural question to ask concerns the possible appearance of a first-order phase transition from a hadronic bag to a deconfined quark–gluon plasma within our framework. This is left for future work.

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Appendix A. Explicit representations for the asymptotic contributions inside the bag

The essential formulae for the basic series $f(s;a;b)$, equation (2.19), in the calculation are the following:

$$\sum_{\nu=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \nu^{2n+1}\left(1+\left(\frac{\nu}{x}\right)^2\right)^{-s} = \frac{1}{2}\frac{n!\Gamma(s-n-1)}{\Gamma(s)}x^{2n+2}$$
$$+(-1)^n 2\int_0^x \mathrm{d}\nu\,\frac{\nu^{2n+1}}{1+\mathrm{e}^{2\pi\nu}}\left(1-\left(\frac{\nu}{x}\right)^2\right)^{-s}$$
$$+(-1)^n 2\cos(\pi s)\int_x^{\infty}\mathrm{d}\nu\,\frac{\nu^{2n+1}}{1+\mathrm{e}^{2\pi\nu}}\left(\left(\frac{\nu}{x}\right)^2-1\right)^{-s} \tag{A.1}$$

$$\sum_{\nu=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \nu^{2n}\left(1+\left(\frac{\nu}{x}\right)^2\right)^{-s} = \frac{1}{2}\frac{\Gamma(n+\frac{1}{2})\Gamma(s-n-\frac{1}{2})}{\Gamma(s)}x^{2n+1}$$
$$-(-1)^n 2\sin(\pi s)\int_x^{\infty}\mathrm{d}\nu\,\frac{\nu^{2n}}{1+\mathrm{e}^{2\pi\nu}}\left(\left(\frac{\nu}{x}\right)^2-1\right)^{-s}. \tag{A.2}$$

Using partial integrations one can obtain representations valid for values of s needed for the $A_i(s)$. One obtains, for example, the following expansions around $s = -\frac{1}{2}$:

$$\sum_{\nu=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \nu^3 \left(1+\left(\frac{\nu}{x}\right)^2\right)^{-s-\frac{3}{2}} = \frac{1}{2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{3}{2})}x^4 - x^2 \int_0^\infty \mathrm{d}\nu \frac{\mathrm{d}}{\mathrm{d}\nu}\left[\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right] \ln|\nu^2 - x^2| + \mathcal{O}(s+\tfrac{1}{2})$$

$$\sum_{\nu=1\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \nu^2 \left(1+\left(\frac{\nu}{x}\right)^2\right)^{-s-\frac{3}{2}} = -\frac{\pi}{2}x^3 + \pi\frac{x^3}{1+\mathrm{e}^{2\pi x}} + \mathcal{O}(s+\tfrac{1}{2})$$

showing clearly that one can obtain quickly convergent integrals, respectively expressions for the effective numerical evaluation of the involved sums. All the particular values that are necessary to give the $A_i(s)$, $i = 1, 2, 3$, explicitly (in addition to the recurrence (2.20)) are listed in appendix B.

The first two leading asymptotic contributions, A_{-1} and A_0 have to be treated in a slightly different way, as has been explained in detail in [45]. For completeness we give the final results

$$\begin{aligned} A_{-1}(s) = {} & \left(\frac{1}{s+\frac{1}{2}} - \ln m^2\right)\left(-\frac{R^3m^4}{12\pi} + \frac{m^2R}{24\pi} + \frac{7}{960\pi R}\right) + \frac{R^3m^4}{24\pi}(1-4\ln 2) \\ & - \frac{m^3R^2}{6} + \frac{m^2R}{24\pi}[2\ln(2mR) - 1] + \frac{7}{960\pi R}[1 + 2\ln(2mR)] \\ & - \frac{2}{\pi R}\int_0^\infty \frac{\mathrm{d}\nu\,\nu}{1+\mathrm{e}^{2\pi\nu}}(\nu^2 - m^2R^2)\ln|\nu^2 - m^2R^2| \\ & - \frac{4m^2R}{\pi}\int_0^\infty \frac{\mathrm{d}\nu\,\nu}{1+\mathrm{e}^{2\pi\nu}}\left(\ln|\nu^2 - m^2R^2| + \frac{\nu}{mR}\ln\left|\frac{mR+\nu}{mR-\nu}\right|\right) \\ & + \frac{m^2R}{2\pi}\ln(1+\mathrm{e}^{-2\pi mR}) - \frac{1}{R}\int_{mR}^\infty \frac{\mathrm{d}\nu\,\nu^2}{1+\mathrm{e}^{2\pi\nu}} - \frac{m^2R}{\pi}\int_0^1 \mathrm{d}y\,\ln(1+\mathrm{e}^{-2\pi mRy}) \end{aligned} \qquad \text{(A.3)}$$

and

$$\begin{aligned} A_0(s) = {} & -\left(\frac{1}{s+\frac{1}{2}} - \ln m^2\right)\left(\frac{1}{48\pi R} + \frac{m^2R}{4\pi}\right) + \frac{m^3R^2}{6} + \frac{m^2R}{\pi}\left[\frac{5}{4} - \frac{1}{2}\ln 2 - \ln(mR)\right] \\ & - \frac{\ln 2}{24\pi R} - \frac{2}{R}\int_{mR}^\infty \frac{\mathrm{d}\nu\,\nu^2}{1+\mathrm{e}^{2\pi\nu}} - m^3R^2\int_0^1 \frac{\mathrm{d}x}{1+\mathrm{e}^{2\pi mR\sqrt{x}}} \\ & + \frac{1}{\pi R}\int_0^\infty \frac{\mathrm{d}\nu\,\nu}{1+\mathrm{e}^{2\pi\nu}}\ln\left|1 - \left(\frac{\nu}{mR}\right)^2\right| \\ & - \frac{m^2R}{2\pi}\int_0^\infty \mathrm{d}\nu\left(\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{1+\mathrm{e}^{2\pi\nu}}\right)\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}\ln|m^2R^2x - \nu^2|. \end{aligned} \qquad \text{(A.4)}$$

This completes the description of our procedure to obtain well-suited representations (for numerical evaluation) of all the A_i that are needed for the calculation of the Casimir energy of the spinor inside the bag.

Appendix B. Full list of constituent terms $f(a; b)$ to be used in addition to the recurrence formula

To simplify the expressions, we shall here use x for mR. In addition to the above recurrence, in order to determine all the $A_i(s)$ explicitly one needs the following $f(a; b)$'s (we shall use the notation $f(a; b) = f(-\frac{1}{2}; a; b)$):

$$
\begin{aligned}
& f(0; \tfrac{1}{2}) = 0 \qquad f(1; \tfrac{1}{2}) = -\tfrac{1}{2}x^2 + \tfrac{1}{24} \\
& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=-\frac{1}{2}} f(s; 0; \tfrac{1}{2}) = -\pi x - 2\pi \int_x^\infty \mathrm{d}\nu \, \frac{1}{1+\mathrm{e}^{2\pi\nu}} \\
& \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=-\frac{1}{2}} f(s; 1; \tfrac{1}{2}) = -\frac{1}{2}x^2 - 2\int_0^\infty \mathrm{d}\nu \, \frac{\nu}{1+\mathrm{e}^{2\pi\nu}} \ln\left|1 - \left(\frac{\nu}{x}\right)^2\right| \\
& f(0; 1) = \frac{x}{2(s+\frac{1}{2})} + x\ln 2 + 2x^2 \int_x^\infty \mathrm{d}\nu \, \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu(1+\mathrm{e}^{2\pi\nu})}\right)\left[\left(\frac{\nu}{x}\right)^2 - 1\right]^{1/2} \\
& f(0; \tfrac{3}{2}) = \frac{\pi x}{2} - \frac{\pi x}{1+\mathrm{e}^{2\pi x}} \\
& f(1; \tfrac{3}{2}) = \frac{x^2}{2(s+\frac{1}{2})} + x^2\ln x + x^2 \int_0^\infty \mathrm{d}\nu \, \left(\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{1+\mathrm{e}^{2\pi\nu}}\right)\ln|\nu^2 - x^2| \\
& f(1; 1) = 2x^2 \int_0^x \mathrm{d}\nu \, \left(\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{1+\mathrm{e}^{2\pi\nu}}\right)\left[1 - \left(\frac{\nu}{x}\right)^2\right]^{1/2} \\
& f(1; 2) = -2x^2 \int_0^x \mathrm{d}\nu \, \left(\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{1+\mathrm{e}^{2\pi\nu}}\right)\left|1 - \left(\frac{\nu}{x}\right)^2\right|^{-1/2} \\
& f(2; 2) = \frac{x^3}{2(s+\frac{1}{2})} + (\ln 2 - 1)x^3 + 2x^4 \int_x^\infty \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu}{1+\mathrm{e}^{2\pi\nu}}\right)\right)\left[\left(\frac{\nu}{x}\right)^2 - 1\right]^{1/2} \\
& f(2; \tfrac{5}{2}) = \frac{\pi}{4}x^3 - \frac{\pi}{2}x^4 \left.\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu}{1+\mathrm{e}^{2\pi\nu}}\right)\right|_{\nu=x} \\
& f(3; \tfrac{5}{2}) = \frac{x^4}{2(s+\frac{1}{2})} + (\ln x - \tfrac{1}{2})x^4 + \frac{x^4}{2}\int_0^\infty \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right)\right]\ln|\nu^2 - x^2| \\
& f(3; 3) = -\frac{2}{3}x^4 \int_0^x \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right)\right]\left[1 - \left(\frac{\nu}{x}\right)^2\right]^{-1/2} \qquad \text{(B.1)} \\
& f(4; 3) = \frac{x^5}{2(s+\frac{1}{2})} + (3\ln 2 - 4)\frac{x^5}{3} + \frac{2x^6}{3}\int_x^\infty \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^3}{1+\mathrm{e}^{2\pi\nu}}\right)\right] \\
& \qquad \times \left[\left(\frac{\nu}{x}\right)^2 - 1\right]^{1/2} \\
& f(4; \tfrac{7}{2}) = \frac{3\pi}{16}x^5 - \frac{\pi}{8}x^6 \left.\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^3}{1+\mathrm{e}^{2\pi\nu}}\right)\right|_{\nu=x} \\
& f(5; \tfrac{7}{2}) = \frac{x^6}{2(s+\frac{1}{2})} + (\ln x - \tfrac{3}{4})x^6 + \frac{x^6}{8}\int_0^\infty \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^4}{1+\mathrm{e}^{2\pi\nu}}\right)\right]\ln|\nu^2 - x^2| \\
& f(5; 4) = -\frac{2x^6}{15}\int_0^x \mathrm{d}\nu \, \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^4}{1+\mathrm{e}^{2\pi\nu}}\right)\right]\left[1 - \left(\frac{\nu}{x}\right)^2\right]^{-1/2}
\end{aligned}
$$

$$f(6;4) = \frac{x^7}{2(s+\frac{1}{2})} + (\ln 2 - \tfrac{23}{15})x^7 + \frac{2x^8}{15}\int_x^\infty \mathrm{d}\nu \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^5}{1+\mathrm{e}^{2\pi\nu}}\right)\right] \times \left[\left(\frac{\nu}{x}\right)^2 - 1\right]^{1/2}$$

$$f(6;\tfrac{9}{2}) = \frac{5\pi}{32}x^7 - \frac{\pi}{48}x^8\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^5}{1+\mathrm{e}^{2\pi\nu}}\right)\Bigg|_{\nu=x}$$

$$f(7;\tfrac{9}{2}) = \frac{x^8}{2(s+\frac{1}{2})} + (\ln x - \tfrac{11}{12})x^8 + \frac{x^8}{48}\int_0^\infty \mathrm{d}\nu \left[\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{1}{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu}\frac{\nu^6}{1+\mathrm{e}^{2\pi\nu}}\right)\right] \times \ln|\nu^2 - x^2|.$$

Using these formulae all the $A_i(s)$ are obtained immediately and, what is important, always in the most suitable fashion for practical evaluation (as explained before).

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